

ONE-DIMENSIONAL GORENSTEIN LOCAL RINGS WITH DECREASING HILBERT FUNCTION

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ABSTRACT. In this paper we solve a problem posed by M.E. Rossi: *Is the Hilbert function of a Gorenstein local ring of dimension one not decreasing?* More precisely, for any integer $h > 1$, $h \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$, we construct infinitely many one-dimensional Gorenstein local rings, included integral domains, reduced and non-reduced rings, whose Hilbert function decreases at level h ; moreover we prove that there are no bounds to the decrease of the Hilbert function. The key tools are numerical semigroup theory, especially some necessary conditions to obtain decreasing Hilbert functions found by the first and the third author, and a construction developed by V. Barucci, M. D'Anna and the second author, that gives a family of quotients of the Rees algebra. Many examples are included.

Introduction

Given a one-dimensional Cohen-Macaulay local ring (R, \mathfrak{m}, k) , let G be its associated graded ring $G = \bigoplus_{h \geq 0} (\mathfrak{m}^h / \mathfrak{m}^{h+1})$ and H_R be the Hilbert function of R , defined as $H_R(h) = H_G(h) = \dim_k (\mathfrak{m}^h / \mathfrak{m}^{h+1})$. The Cohen-Macaulayness of G and the behaviour of the Hilbert function are classic topics in local algebra. Starting from 1970s with the basic results of J.D. Sally [26], [27], [28], many authors have contributed to these themes; for instance we recall J. Elias [16], M.E. Rossi and G. Valla [25] and Rossi's survey [24]. It is well-known that if G is Cohen-Macaulay, the function H_R is non-decreasing. On the other hand, when $\text{depth}(G) = 0$, H_R can decrease, i.e. $H_R(h-1) > H_R(h)$ for some h ; in this case we say that H_R decreases at level h and that R has decreasing Hilbert function. When R is Gorenstein, M.E. Rossi asked in [24, Problem 4.9] if H_R is always non-decreasing and in the last decade several authors found partial positive answers to this problem, especially in the case of numerical semigroup rings:

- In [2] F. Arslan and P. Mete, for large families of complete intersection rings and the Gorenstein numerical semigroup rings with embedding dimension 4, under some arithmetical conditions;
- In [3] F. Arslan, P. Mete and M. Şahin, for infinitely many families of Gorenstein rings obtained by introducing the notion of nice gluing of numerical semigroups;
- In [22] D.P. Patil and the third author, for the rings associated with balanced numerical semigroups with embedding dimension 4;
- In [4] F. Arslan, N. Sipahi and N. Şahin, for other 4-generated Gorenstein numerical

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semigroup rings constructed by non-nice gluing;

- In [19] R. Jafari and S. Zarzuela Armengou, for some families of numerical Gorenstein semigroup rings through the concept of extension;
- In [1] F. Arslan, A. Katsabekis, and M. Nalbandiyan, for other families of Gorenstein 4-generated numerical semigroup rings;
- In [21] the first and the third author, for numerical semigroup rings such that $\nu \geq e - 4$, where ν and e denote respectively the embedding dimension and the multiplicity of R .

In this paper we show that Rossi's problem has negative answer, by constructing, among others, explicit examples of Gorenstein numerical semigroup rings with decreasing Hilbert function. They are particular rings of a family introduced and studied by V. Barucci, M. D'Anna and the second author in [6] and [7] to provide a unified approach to Nagata's idealization and amalgamated duplication. Given a commutative ring R and an ideal I , for any $a, b \in R$ the rings $R(I)_{a,b}$ are defined as suitable quotients of the Rees algebra of I . These have many good properties, in particular, if R is a one-dimensional local ring, so is $R(I)_{a,b}$. If R is Cohen-Macaulay, another important fact is that $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal of R ; in this case, when R is almost Gorenstein, we prove that the Hilbert function of $R(I)_{a,b}$ depends only on the Cohen-Macaulay type and the Hilbert function of R . The crucial result is that if R is an almost Gorenstein ring with $H_R(h - 2) > H_R(h)$ for some $h \geq 3$ and I is a canonical ideal of R , then $R(I)_{a,b}$ is a one-dimensional Gorenstein local ring with Hilbert function decreasing at level h .

We find such rings through numerical semigroup theory. A numerical semigroup S is a submonoid of the natural numbers that has finite complement in \mathbb{N} ; if S is generated by $s_0, \dots, s_{\nu-1}$ and k is a field, then $k[[S]] := k[[t^{s_0}, \dots, t^{s_{\nu-1}}]]$ is the numerical semigroup ring associated with S and many properties of $k[[S]]$, such as the Hilbert function, are still contained in S . In this context $k[[S]]$ is almost Gorenstein if and only if S is a so-called almost symmetric semigroup.

Hence to achieve our results we look for almost symmetric semigroups with decreasing Hilbert function. If e and ν are the multiplicity and embedding dimension of S (or equivalently of $k[[S]]$) we first show that we need $e - \nu \geq 4$. By using a result of [21] and a theorem of H. Nari [20] we give an explicit construction of a family of almost symmetric semigroups with the required properties. We also show other examples with the above properties. In conclusion we prove that for any integers $m \geq 1$ and $h > 1, h \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$, there exist infinitely many non-isomorphic one-dimensional Gorenstein local rings R such that $H_R(h - 1) - H_R(h) > m$; this class always contains numerical semigroup rings, non-reduced rings, and reduced rings that are not integral domains.

We include several examples in the numerical semigroup case. In fact if R is a numerical semigroup ring and $b = t^m$, with m odd, the ring $R(I)_{0,-b}$ is isomorphic to the numerical semigroup ring associated with the numerical duplication, a construction introduced and studied by M. D'Anna and the second author in [14].

The structure of the paper is the following. In the first section we introduce the family $R(I)_{a,b}$ and show how to reduce the problem to find a suitable almost Gorenstein ring, see Corollary 1.5. In Section 2 we describe a procedure that gives infinitely many almost Gorenstein semigroup rings satisfying the desired properties, see Construction 2.6 and Theorem

2.9. In Section 3 we prove the main result, see Theorem 3.3; moreover we give explicit examples of one-dimensional Gorenstein local semigroup rings with decreasing Hilbert function and other interesting examples based on the above constructions; see e.g. Example 3.4 with Hilbert function $[1, 53, 54, 54, 53, 53, 56, 59, 61, 63, 64 \rightarrow]$, Example 3.9 with Hilbert function decreasing at many levels, and Example 3.10 for a ring with smaller multiplicity and embedding dimension. Finally the appendix contains the technical results needed to prove Theorem 2.9.

1. Reduction to the almost Gorenstein case

Let R be a commutative ring with identity and let I be a proper ideal of R . The Rees algebra of I is the ring $\mathcal{R}_+ := \bigoplus_{n \geq 0} I^n t^n \subseteq R[t]$. Consider the ideal $(t^2 + at + b)$ of $R[t]$, where a and b are elements of R , and let $I^2(t^2 + at + b)$ denote its contraction to the Rees algebra. In [6] it is introduced and studied the following family of rings

$$R(I)_{a,b} := \frac{\mathcal{R}_+}{I^2(t^2 + at + b)}.$$

One aim of this construction was to provide a unified approach to Nagata's idealization (see [5]) and amalgamated duplication (see [11] and [13]), which are isomorphic to $R(I)_{0,0}$ and $R(I)_{-1,0}$ respectively. In fact in [6] and [7] it is proved that several properties of the family are independent of a and b . In particular R is local if and only if $R(I)_{a,b}$ is local, R and $R(I)_{a,b}$ have the same dimension and $R(I)_{a,b}$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and I is a maximal Cohen-Macaulay module; further, in the last case if I contains a regular element, $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal of R .

We are interested in the Hilbert function of these rings. In [6, Proposition 2.3] it is stated that their Hilbert functions do not depend on a and b , but actually in the proof it is shown more and then we restate that proposition in the version we need:

Proposition 1.1. *If (R, \mathfrak{m}) is a local ring and I is an ideal of R , then for any $h \geq 2$ $H_{R(I)_{a,b}}(h) = H_R(h) + \ell_R(I\mathfrak{m}^{h-1}/I\mathfrak{m}^h)$, where ℓ_R denotes the length as R -module. In particular it is independent of a and b .*

We are interested in the case in which R is a one-dimensional Cohen-Macaulay local ring and I is a canonical ideal of R . In this case we can easily compute $\ell_R(I\mathfrak{m}^{h-1}/I\mathfrak{m}^h)$, under an extra hypothesis on R : almost Gorensteinness. Following [8] and [18] we recall the definition in the one-dimensional case:

Definition 1.2. Let (R, \mathfrak{m}) be a one-dimensional Cohen-Macaulay local ring with a canonical module ω_R such that $R \subseteq \omega_R \subseteq \overline{R}$. Then R is said to be *almost Gorenstein* if $\mathfrak{m}\omega_R = \mathfrak{m}$.

From now on we assume that R is one-dimensional. In the setting of the previous definition, chosen a regular element $a \in R$ such that $a\omega_R \subset R$, the ideal $I = a\omega_R$ is a canonical ideal of R and all canonical ideals of R can be obtained in this way (see e.g. [18, Corollary 2.8]). If R is an almost Gorenstein ring and I is a canonical ideal of R , for any $h \geq 2$ we have

$$(1.3) \quad \ell_R\left(\frac{I\mathfrak{m}^{h-1}}{I\mathfrak{m}^h}\right) = \ell_R\left(\frac{a\omega_R\mathfrak{m}^{h-1}}{a\omega_R\mathfrak{m}^h}\right) = \ell_R\left(\frac{a\mathfrak{m}^{h-1}}{a\mathfrak{m}^h}\right) = \ell_R\left(\frac{\mathfrak{m}^{h-1}}{\mathfrak{m}^h}\right) = H_R(h-1).$$

Therefore we get the following:

Proposition 1.4. *Let R be an almost Gorenstein ring and let I be a canonical ideal of R . Let $\nu(R)$ and $t(R)$ denote the embedding dimension and the Cohen-Macaulay type of R respectively. Then the Hilbert function of $R(I)_{a,b}$ is:*

$$\begin{aligned} H_{R(I)_{a,b}}(0) &= 1 \\ H_{R(I)_{a,b}}(1) &= \nu(R) + t(R) \\ H_{R(I)_{a,b}}(h) &= H_R(h) + H_R(h-1) \quad \text{if } h \geq 2. \end{aligned}$$

Proof. By Proposition 1.1 and (1.3) we only need to show the statement about $H_{R(I)_{a,b}}(1)$: actually this equality is true even if R is not almost Gorenstein. In fact, since the minimal number of generators of a canonical module is the Cohen-Macaulay type of the ring (see e.g. [9, Proposition 3.3.11]), it is easy to deduce the thesis from Proposition 1.1. \square

Corollary 1.5. *Let R be an almost Gorenstein ring and let I be a canonical ideal of R . Then $R(I)_{a,b}$ is a Gorenstein ring for any choice of $a, b \in R$. Further $H_{R(I)_{a,b}}(1) - H_{R(I)_{a,b}}(2) = t(R) - H_R(2)$ and $H_{R(I)_{a,b}}(h-1) - H_{R(I)_{a,b}}(h) = H_R(h-2) - H_R(h)$ for any $h \geq 3$.*

Proof. Since I is a canonical ideal, $R(I)_{a,b}$ is a Gorenstein ring by [6, Corollary 3.3]. Moreover, by the previous proposition, for any $h \geq 3$ we have

$$H_{R(I)_{a,b}}(h-1) - H_{R(I)_{a,b}}(h) = H_R(h-1) + H_R(h-2) - H_R(h) - H_R(h-1) = H_R(h-2) - H_R(h).$$

The first formula can be found in the same way, since $H_R(1) = \nu(R)$. \square

Remark 1.6. In this paper we are interested in *negative results*, anyway it is clear that Proposition 1.1 can be also used to get *positive results*. A local ring S is said to have minimal multiplicity if its multiplicity is $1 + \text{codim } S$. In [23, Theorem 1.1] it is proved that if S is a two-dimensional Cohen-Macaulay local ring with minimal multiplicity and (R, \mathfrak{m}) is a one-dimensional Cohen-Macaulay local ring which is a quotient of S , then every maximal Cohen-Macaulay R -module M has non-decreasing Hilbert function. Here the h -th value of the Hilbert function of M is defined as the length of $M\mathfrak{m}^h/M\mathfrak{m}^{h+1}$. Therefore, Proposition 1.1 implies that, under the above hypothesis, $R(I)_{a,b}$ has non-decreasing Hilbert function for any maximal Cohen-Macaulay ideal I ; in particular, if I is a canonical ideal, $R(I)_{a,b}$ is a one-dimensional Gorenstein local ring with non-decreasing Hilbert function for all $a, b \in R$. See [23] for several explicit cases in which the above hypothesis hold.

2. Construction of almost symmetric semigroups

In order to obtain Gorenstein local rings with decreasing Hilbert function, by Corollary 1.5 it is enough to find almost Gorenstein rings R such that $H_R(h-2) > H_R(h)$; in this section we construct infinitely many semigroup rings verifying these conditions.

First, we briefly recall some definitions and properties about numerical semigroup theory that we need. A *numerical semigroup* S is a submonoid of the natural numbers such that $|\mathbb{N} \setminus S| < \infty$. The maximum element of $\mathbb{N} \setminus S$ is called *Frobenius number* of S and it will be denoted by $f(S)$. If S is generated by $n_0 \leq n_1 \leq \dots \leq n_{\nu-1}$, we write $S = \langle n_0, \dots, n_{\nu-1} \rangle$. It is well-known that a numerical semigroup has an unique minimal system of generators and its cardinality is the *embedding dimension* ν of S . The smallest non-zero element of S is n_0 ; it is called *multiplicity* of S and we will denote it by $e(S)$ or simply e , if the

semigroup is clear from the context. A *numerical semigroup ring* is a local ring of the form $k[[S]] := k[[t^{n_0}, t^{n_1}, \dots, t^{n_{\nu-1}}]]$, where $S = \langle n_0, n_1, \dots, n_{\nu-1} \rangle$ is a numerical semigroup and k a field. A *relative ideal* E of a numerical semigroup is a subset of \mathbb{Z} such that there exists $x \in \mathbb{N}$ for which $x + E \subseteq S$ and $E + S \subseteq E$; if E is contained in S we say that E is a (proper) ideal of S . An example of ideal is the *maximal ideal* $M = M(S) := S \setminus \{0\} = v(\mathfrak{m})$, where $\mathfrak{m} = (t^{n_0}, t^{n_1}, \dots, t^{n_{\nu-1}})$ and $v : k((t)) \rightarrow \mathbb{Z} \cup \{\infty\}$ is the usual valuation. An example of a relative ideal is the *standard canonical ideal* $K(S) := \{x \in \mathbb{N} \mid f(S) - x \notin S\}$; more generally, we call *canonical ideals* all the relative ideals $K(S) + z$ for any $z \in \mathbb{Z}$.

The properties of a semigroup ring are strictly related to those of the associated numerical semigroup. In particular: $H_{K[[S]]}(h) = |v(\mathfrak{m}^h) \setminus v(\mathfrak{m}^{h+1})| = |hM \setminus (h+1)M|$ for each $h \geq 1$. We shall denote this function by H_S and its values by $[1, \nu(S), H_S(2), \dots, e, \rightarrow]$. It is well-known that:

- $k[[S]]$ is Gorenstein if and only if $K(S) = S$; in this case S is said to be *symmetric*;
- $k[[S]]$ is almost Gorenstein if and only if $M + K(S) = M$; in this case S is said to be *almost symmetric*.

Definition 2.1. Let S be a numerical semigroup.

- i. If s is an element of S , the *order* of s is $\text{ord}(s) := \max\{i \mid s \in iM\}$.
- ii. The *Apéry set* of S is $\text{Ap}(S) := \{s \in S \mid s - e \notin S\}$, shortly Ap ; it has cardinality e .
- iii. $\text{Ap}_k := \{s \in \text{Ap} \mid \text{ord}(s) = k\}$.
- iv. $D_h := \{s \in S \mid \text{ord}(s) = h - 1 \text{ and } \text{ord}(s + e) > h\}$, $D_h^t := \{s \in D_h \mid \text{ord}(s + e) = t\}$.
- v. $C_k := \{s \in S \mid \text{ord}(s) = k \text{ and } s - e \notin (k - 1)M\}$.

We notice that $C_k = \text{Ap}_k \cup \{\cup_h (D_h^k + e) \mid 2 \leq h \leq k - 1\}$ and $H_S(k - 1) - H_S(k) = |D_k| - |C_k|$, see for instance [10] and [22].

The following theorem of H. Nari characterizes the almost symmetric numerical semigroups by means of their Apéry sets. First we recall that a *pseudo-Frobenius number* of S is an integer $x \in \mathbb{Z} \setminus S$ such that $x + s \in S$ for any $s \in M$. We denote the set of pseudo-Frobenius number of S by $\text{PF}(S)$; it is straightforward to see that $f(S) \in \text{PF}(S)$. The cardinality of $\text{PF}(S)$ is the *type* of S and it will be denoted by $t(S)$; it is well-known that $t(S) = t(k[[S]])$.

Theorem 2.2. [20, Theorem 2.4] *Let S be a numerical semigroup. Set $\text{Ap} = A \cup B$, where $A := \{0 < \alpha_1 < \dots < \alpha_m\}$, $B := \{\beta_1 < \dots < \beta_{t(S)-1}\}$, with $m = e - t(S)$, and $\text{PF}(S) = \{\beta_i - e \mid 1 \leq i \leq t(S) - 1\} \cup \{\alpha_m - e = f(S)\}$. The following conditions are equivalent:*

- i. S is almost symmetric;
- ii. $\alpha_i + \alpha_{m-i} = \alpha_m$ for all $i \in \{1, 2, \dots, m - 1\}$ and $\beta_j + \beta_{t(S)-j} = \alpha_m + e$ for all $j \in \{1, 2, \dots, t(S) - 1\}$.

Since we are looking for a semigroup with decreasing Hilbert function, we need that $|\text{Ap}_2| \geq 3$, by [12, Corollary 3.11]; then we focus on the simpler case, $|\text{Ap}_2| = 3$.

Proposition 2.3. *Assume $|\text{Ap}_2| = 3$, $\text{Ap}_k = \emptyset$ for all $k \geq 3$ and H_S decreasing. Then S cannot be almost symmetric.*

Proof. The assumptions on the Apéry set imply that $\nu = e - 3$. Since H_S decreases, by [21, Theorem 4.2.3] there exist $n_1 \neq n_2 \in \text{Ap}_1$ such that $\text{Ap}_2 = \{2n_1, n_1 + n_2, 2n_2\}$ and we

assume $n_1 < n_2$. The element $n_1 - e$ is not a pseudo-Frobenius number, because $2n_1 - e \notin S$; therefore if S is almost symmetric, with the notation of the previous theorem, A is non-empty. It follows that $\text{ord}(\alpha_m) > 1$ and so $\alpha_m = 2n_2$. On the other hand by [21, Proposition 4.3.1] we have $3n_2 - e \in Ap$, that is a contradiction because $3n_2 - e > 2n_2$. \square

According to the above proposition, we consider the next case $|Ap_3| = 1$. In this context the following proposition holds:

Proposition 2.4. [21, Proposition 3.4] *Assume $|Ap_2| = 3$, $|Ap_3| = 1$ and H_S decreasing. Let $\ell = \min\{h \mid H_S \text{ decreases at level } h\}$ and let $d = \max\{\text{ord}(\sigma) \mid \sigma \in Ap\}$. Then $\ell \leq d$ and there exist $n_1, n_2 \in Ap_1$ such that for $2 \leq h \leq \ell$*

$$C_h = \{hn_1, (h-1)n_1 + n_2, \dots, n_1 + (h-1)n_2, hn_2\} = Ap_h \cup (D_{h-1} + e)$$

$$D_\ell + e = \{(d+1)n_1, \ell n_1 + n_2, (\ell-1)n_1 + 2n_2, \dots, (\ell+1)n_2\}.$$

Further, if $(\ell, d) \neq (3, 3)$, then $Ap_k = kn_1$, for $3 \leq k \leq d$.

The next proposition shows that, in the setting of the previous one, we only need to find an almost symmetric semigroup with decreasing Hilbert function. This is not true in general, for instance the numerical semigroup $S = \langle 30, 35, 42, 47, 108, 110, 113, 118, 122, 127, 134, 139 \rangle$ is almost symmetric and its Hilbert function is $H_S = [1, 12, 17, 16, 25, 30 \rightarrow]$. Therefore H_S decreases, but $H_S(h-2) \leq H_S(h)$ for any $h \geq 2$.

Proposition 2.5. *Assume that $|Ap_2| = 3$, $|Ap_3| = 1$, and H_S is decreasing. Let ℓ be the minimum level in which the Hilbert function of S decreases:*

- i. If $\ell \geq 3$, then $H_S(h) = H_S(\ell-1)$ for all $h \in [1, \ell-1]$. Further, $H_S(\ell-2) - H_S(\ell) = 1$.*
- ii. If S is almost symmetric then $\alpha_m = dn_1$ and $kn_1 \in A$ for any $k \leq d$ (see Theorem 2.2 and Proposition 2.4 for the notation).*
- iii. If S is almost symmetric, then $\ell \geq 3$.*

Proof. *i.* By Proposition 2.4, if $2 < h < \ell$ we have $C_h = (D_{h-1} + e) \cup Ap_h$ and then $|D_{h-1}| = |C_h| - 1$, since $|Ap_h| = 1$. Moreover, in this range, we have $|C_h| = |C_{h-1}| + 1$, because of the previous proposition. Hence for any $h = 2, \dots, \ell-1$ we have

$$H_S(h-1) - H_S(h) = |D_h| - |C_h| = |C_{h+1}| - 1 - (|C_{h+1}| - 1) = 0.$$

Consequently we get $H_S(1) = H_S(2) = \dots = H_S(\ell-1)$. As for the last part of the statement it is enough to note that, by the previous proposition, we have

$$H_S(\ell-1) - H_S(\ell) = |D_\ell| - |C_\ell| = \ell + 2 - (\ell + 1) = 1.$$

- ii.* Of course α_m is the greatest element of the Apéry set and in our case it can be either dn_1 or $2n_2$. If $\alpha_m = 2n_2$, then there would exist $n \in Ap$, such that $(d-1)n_1 + n = 2n_2$, but it is impossible, since $d \geq 3$ implies $\text{ord}(2n_2) \geq 3$. Clearly $kn_1 \in A$, because $kn_1 + (d-k)n_1 = dn_1$.
- iii.* Assume $\ell = 2$. By Proposition 2.4, $(d+1)n_1 - e \in D_2$ and so $\text{ord}((d+1)n_1 - e) = 1$; consequently, if S is almost symmetric, there exists $n \in Ap$ such that $(d+1)n_1 - e + n = dn_1 + ke$, with either $k = 0$ or $k = 1$. Hence $n_1 + n = (1+k)e \leq 2e$, impossible. \square

The next construction is based on Proposition 2.4 and indeed we will prove that it defines almost symmetric numerical semigroups with decreasing Hilbert functions satisfying the assumptions of Proposition 2.4.

Construction 2.6. Let $\ell \in \mathbb{N}, \ell \geq 4, \ell \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$, let

$$\begin{aligned} e &:= \ell^2 + 3\ell + 4 \\ \left[\begin{array}{ll} n_1 := \ell^2 + 5\ell + 3 &= e + (2\ell - 1), \quad n_2 := 2\ell^2 + 3\ell - 2 = e + (\ell^2 - 6), & \text{if } \ell \text{ odd} \\ n_1 := \ell^2 + 4\ell + 1 &= e + (\ell - 3), \quad n_2 := 2\ell^2 + 2\ell - 2 = e + (\ell^2 - \ell - 6), & \text{if } \ell \text{ even} \end{array} \right. \end{aligned}$$

and let S be the semigroup generated by the subset $\Gamma \subseteq \mathbb{N}$

$$\Gamma = \{e, n_1, n_2\} \cup \{t_1, t_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\} \setminus \{n_1 + n_2, 2n_2\}$$

where:

$$\begin{aligned} \{s_{p,q}\} &= \{pn_1 + qn_2 - (p+q-2)e \mid 0 \leq p \leq \ell, 1 \leq q \leq \ell+1, 2 \leq p+q \leq \ell+1\} \\ \{r_{p,q}\} &= \{\ell n_1 + e - s_{p,q} \mid 2 \leq p+q \leq \ell+1, p \geq 1, q \geq 1\} \\ t_1 &= (\ell+1)n_1 - (\ell-1)e, \\ t_2 &= \ell n_1 + e - t_1 = (\ell-1)e - n_1 \\ |\{s_{p,q}\}| &= (\ell^2 + 3\ell)/2 := u, \quad |\{r_{p,q}\}| = (\ell^2 + \ell)/2, \quad |\Gamma| = |\{s_{p,q}\}| + |\{r_{p,q}\}| + 3 = e - \ell - 1 \\ (\ell-1)e &= -\ell n_1 + (\ell+2)n_2 \end{aligned}$$

We note that the elements $\{e, n_1, n_2, t_1\} \cup \{s_{p,q}\}$ are given to obtain the structure of S required in Proposition 2.4, with $d = \ell$, while the elements $\{r_{p,q}\} \cup \{t_2\}$, impose the almost symmetry of S following Theorem 2.2 (with $\alpha_m = \ell n_1$, and $B \supseteq \{t_1\} \cup \{s_{p,q}\} \setminus \{s_{0,\ell+1}\}$). In this construction, for $s \in \{qn_2 - (q-2)e, 2 \leq q \leq \ell+1\} \cup \{n_2\} \cup \{pn_1, 1 \leq p \leq \ell\}$, we don't need to add the corresponding $\ell n_1 + e - s$, or $\ell n_1 - s$, because such element is already inside S (see Lemma 4.2. i).

Remark 2.7. Looking for an almost symmetric semigroup S satisfying Proposition 2.4, with $d = \ell$, it is natural to impose $e \geq \ell^2 + 3\ell + 4$. In fact the first idea to construct this semigroup is to impose that a system of generators of S contains the set

$$\{e, n_1, n_2\} \cup \{\ell n_1 - n_2\} \cup \{t_1, t_2\} \cup \{s_{p,q}\} \cup \{\alpha_m + e - s, s \in \{s_{p,q}\}\} \setminus \{n_1 + n_2, 2n_2\}.$$

By counting the number of conditions for a given ℓ , we get

$$e(S) \geq 4 + 2 \frac{\ell^2 + 3\ell}{2} = \ell^2 + 3\ell + 4.$$

Further, following Proposition 2.4, we need

$$Ap = Ap_1 \cup \{2n_1, n_1 + n_2, 2n_2\} \cup \{kn_1, 3 \leq k \leq \ell\}$$

thus the embedding dimension of S must be $\nu = e(S) - \ell - 1$. If $|\{r_{p,q}\}| = |\{s_{p,q}\}| - \ell$ and $\ell n_1 - n_2 \in \{s_{p,q}\}$, we could fix the minimal value $e = \ell^2 + 3\ell + 4$, for the multiplicity. This happens if we define e, n_1 , and n_2 as in Construction 2.6. In fact this choice gives the basic relation

$$\ell n_1 = (\ell+2)n_2 - (\ell-1)e$$

which assures that, for $2 \leq q \leq \ell$, $\ell n_1 + e - s_{0,q} \in \{s_{p,q}\}$ and so it reduces the number of independent conditions to $2|\{s_{p,q}\}| - \ell + 3$ (see Lemma 4.2).

We give some examples before to prove of the exactness of the construction.

Example 2.8. In this example we show the almost symmetric semigroups constructed by means of the above algorithm for $\ell = 4, 5$ and the Hilbert function H_S of $R = k[[S]]$.

$\ell = 4$. The semigroup S is minimally generated by $\{32, 33 = n_1, 38 = n_2, 69, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95\}$. Moreover

$Ap_2 = \{66, 71, 76\}$, $Ap_3 = \{99 = 3n_1\}$, $Ap_4 = \{132 = \ell n_1\}$, $\text{PF}(S) = \{37, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 63, 100\}$.

$$H_S = [1, 27, 27, 27, 26, 27, 29, 30, 31, 32 \rightarrow]$$

$\ell=5$. The minimal generating system of S is $\{44, 53 = n_1, 63 = n_2, 117, 125, 127, 134, 135, 136, 137, 142, 143, 144, 145, 146, 147, 152, 153, 154, 155, 156, 157, 162, 163, 164, 165, 166, 167, 172, 173, 174, 175, 182, 183, 184, 192, 193, 202\}$.

$Ap_2 = \{106, 116, 126\}$, $Ap_3 = \{159\}$, $Ap_4 = \{212\}$, $Ap_5 = \{265\}$ $\text{PF}(S) = \{72, 73, 81, 82, 83, 90, 91, 92, 93, 98, 99, 100, 101, 102, 103, 108, 109, 110, 111, 112, 113, 118, 119, 120, 121, 122, 123, 128, 129, 130, 131, 138, 139, 140, 148, 149, 221\}$.

$$H_S = [1, 38, 38, 38, 38, 37, 44 \rightarrow]$$

To validate Construction 2.6, we need some technical lemmas proved in the appendix.

Theorem 2.9. *With the assumptions of Construction 2.6:*

i. *The ring $R = k[[S]]$ is almost Gorenstein with Hilbert function decreasing at level ℓ :*

$$H_R = [1, \nu, \nu, \dots, \nu, \nu - 1, H_R(\ell + 1) \dots].$$

ii. *The embedding dimension of R is $\nu(R) = e - (\ell + 1) = \ell^2 + 2\ell + 3$.*

iii. *The Cohen-Macaulay type of R is $t(R) = \nu(R) - 1 = \ell^2 + 2\ell + 2$.*

Proof. i. By Proposition 4.5, we know the Apéry set of S and the subsets A, B of Theorem 2.2. Then S is almost symmetric, since the elements of A and B verify the conditions of Theorem 2.2, respectively by Lemma 4.2.i and by the definition of $\{r_{p,q}\}$.

Now we show that the Hilbert function of $k[[S]]$ decreases at level ℓ : we shall prove that

- for each $k \in [2, \ell + 1]$, $C_k = \{kn_1, (k-1)n_1 + n_2, (k-2)n_1 + 2n_2, \dots, kn_2\}$, $|C_k| = k + 1$
- for each $k \in [2, \ell - 1]$, $D_k + e = \{kn_1 + n_2, (k-1)n_1 + 2n_2, \dots, (k+1)n_2\} = C_{k+1} \setminus \{(k+1)n_1\}$, $|D_k| = k + 1$
- $D_\ell + e = \{(\ell + 1)n_1, \ell n_1 + n_2, (\ell - 1)n_1 + 2n_2, \dots, (\ell + 1)n_2\}$, $|D_\ell| = \ell + 2$.

Hence the thesis follows, recalling that for each $k \geq 2$: $H(k) = H(k-1) + |C_k| - |D_k|$.

Let $s = an_1 + bn_2$, with $a + b = k \in [3, \ell + 1]$ and $a < k$, if $k \neq \ell + 1$. First we prove that if $an_1 + bn_2 = d + e$, with $d \in D_h$, then $\text{ord}(an_1 + bn_2) = h + 1$ (1)

In fact, if $\text{ord}(an_1 + bn_2) = k' = h + p$, with $p \geq 2$, we know, by [21, Proposition 2.2.1], that given a maximal expression $an_1 + bn_2 = \sum a_i n_i$, with $n_i \in Ap_1$, $\sum a_i = k'$, for any $y = \sum b_i n_i$, $0 \leq b_i \leq a_i$, $\sum b_i = q \leq p + 1$, then $y \in Ap_q$. Since $k' > 3$ and $Ap_3 = \{3n_1\}$, this would imply $an_1 + bn_2 = k'n_1$, impossible by Lemma 4.4.ii. Hence $an_1 + bn_2 \in C_{h+1} \cap (D_h + e)$.

By definition, $C_2 = Ap_2$ and by Proposition 4.5, $D_2 \supseteq \{2n_1 + n_2 - e, n_1 + 2n_2 - e, 3n_2 - e\}$. Then $D_2 = \{2n_1 + n_2 - e, n_1 + 2n_2 - e, 3n_2 - e\}$, otherwise the Hilbert function decreases at level 2, impossible by Proposition 2.5.iii. Hence $C_3 = (D_2 + e) \cup \{3n_1\}$.

Now we proceed by induction on k . First we recall that, if $x \in C_k$ has maximal representation $x = \sum a_i n_i$, $\sum a_i = k$, $n_i \in Ap_1$ and $y = \sum b_i n_i$, $0 \leq b_i \leq a_i$ with $\sum b_i = h$, then $y \in C_h$ by [21, Proposition 1.4.1]. In our case $C_2 = \{2n_1, n_1 + n_2, 2n_2\}$, hence $C_k \subseteq \{an_1 + bn_2 \mid a + b = k\}$. Assume $3 \leq k \leq \ell$ and the thesis true for $k - 1$. Therefore we know the structures of C_2, \dots, C_k , D_2, \dots, D_{k-1} . Let $an_1 + bn_2$ with $a + b = k + 1 \in [4, \ell + 1]$ and $a < k + 1$, if $k \neq \ell$. Then, by Lemma 2.6, $s_{a,b} = an_1 + bn_2 - (k-1)e \in Ap_1$ is such that $\text{ord}(s_{a,b} + (k-1)e) \geq k + 1$. Moreover, since $s_{a,b} + e \notin D_2$, we know that its order is 2; hence

there exists $r \in [1, k-2]$ such that $\text{ord}(s_{a,b} + re) = r+1$ and $\text{ord}(s_{a,b} + (r+1)e) > r+2$ i.e. $s_{a,b} + re \in D_{r+2}$, with $r+2 \leq k$. If $r+2 < k$, by induction there would exist $a'n_1 + b'n_2 - e \in D_{r+2}$ such that $s_{a,b} + re = a'n_1 + b'n_2 - e = s_{a',b'} + (a' + b' - 3)e$: impossible because $s_{a,b}$ and $s_{a',b'}$ have distinct residues (*mod* e), by Lemma 4.4.ii. Hence $r = k-2$, $an_1 + bn_2 - e \in D_k$ and $an_1 + bn_2 \in C_{k+1}$ by (1). This proves *i*.

ii. Since there are $\ell + 1$ elements of the Apéry set with order greater than 1, we have $\nu(R) = |Ap_1| + 1 = e - 1 - (\ell + 1) + 1 = \ell^2 + 2\ell + 3$.

iii. The Cohen-Macaulay type of $k[[S]]$ is the cardinality of the Pseudo-Frobenius set of S :

$$t(R) = |B| + 1 = \frac{\ell^2 + 3\ell}{2} - 1 + \frac{\ell^2 + \ell}{2} + 3 = \ell^2 + 2\ell + 2. \quad \square$$

Without using Construction 2.6, but by similar techniques, it is possible to construct other almost symmetric semigroups such that $H_S(h-1) > H_S(h)$, even if $h = 2, 3$, as shown in the first two semigroups of the next example. Moreover the last one is another almost symmetric semigroup with $|Ap_2| = 3$, $|Ap_3| = 1$ and decreasing Hilbert function.

Example 2.10. *i.* The numerical semigroup $S = \langle 33, 41, 42, 46, 86, 90, 91, 95, 96, 97, 98, 100, 101, 103, 104, 105, 106, 109, 110, 111, 113, 114, 118, 122 \rangle$ is almost symmetric and its Hilbert function $[1, 24, 23, 23, 31, 33 \rightarrow]$ decreases at level 2. Moreover

$$Ap_2 = \{82, 83, 84, 87, 88, 92, 127\}, \quad Ap_3 = \{126\}, \quad Ap_4 = \{168\}, \quad Ap_k = \emptyset \quad \text{if } k \geq 5.$$

ii. The numerical semigroup $S = \langle 32, 33, 38, 58, 59, 60, 61, 62, 63, 67, 68, 69, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88 \rangle$ is almost symmetric with Hilbert function $[1, 28, 28, 27, 27, 29, 30, 31, 32 \rightarrow]$ decreasing at level 3 and

$$Ap_2 = \{66, 71, 76, 121\}, \quad Ap_k = \emptyset \quad \text{if } k \geq 3.$$

iii. The numerical semigroup $S = \langle 30, 33, 37, 64, 68, 71, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92, 94, 95, 98, 101 \rangle$ is almost symmetric with Hilbert function $[1, 25, 25, 25, 24, 27, 28, 29, 30 \rightarrow]$

$$Ap_2 = \{66, 70, 74\}, \quad Ap_3 = \{99\}, \quad Ap_4(S) = \{132\}, \quad Ap_k = \emptyset \quad \text{if } k \geq 5.$$

3. The Gorenstein case

In this section we give explicit examples of local one-dimensional Gorenstein rings with decreasing Hilbert function and other interesting examples. Several computations are performed by using the GAP system [17] and, in particular, the NumericalSgps package [15].

Luckily, if R is a numerical semigroup ring and $b = t^m \in R$, with m odd, then $R(I)_{0,-b}$ is a numerical semigroup ring and it is exactly the ring associated with the so-called numerical duplication. Anyway we note that in general, for other choices of a and b , the ring $R(I)_{a,b}$ is not a numerical semigroup. For example if $a = -1$ and $b = 0$ it is isomorphic to the amalgamated duplication that, in this case, is reduced but not a domain; while $R(I)_{0,0}$ is isomorphic to the idealization and then it is not reduced. In this section we describe the particular case of the numerical duplication, that is probably the easiest case; we show the most notable and simple examples among the various we have constructed.

Let S be a numerical semigroup, $b \in S$ be an odd integer and E be a proper ideal of S . The *numerical duplication* of S with respect to E and b , introduced in [14], is the numerical

semigroup

$$S \bowtie^b E := \{2 \cdot S\} \cup \{2 \cdot E + b\},$$

where $2 \cdot X = \{2x \mid x \in X\}$ for any set X ; we note that $2 \cdot X$ is different from $2X = X + X$.

As mentioned above, if $R = k[[S]]$ is a numerical semigroup ring and $b = t^m \in R$ with m odd, it is proved in [6, Theorem 3.4] that

$$R(I)_{0,-b} \text{ is isomorphic to } k[[S \bowtie^m E]],$$

where $E := v(I)$ is the valuation of I , see [6] for more detail. We recall that I is a canonical ideal of R if and only if $v(I)$ is a proper canonical ideal of S ; hence $S \bowtie^b E$ is symmetric if and only if E is a canonical ideal, see also [14, Proposition 3.1] for a simpler proof.

It is easy to compute the generators of $S \bowtie^b E$, in fact if $G(S) = \{n_1, \dots, n_r\}$ is the set of the minimal generators of S and E is generated, as ideal, by $\{m_1, \dots, m_s\}$, then

$$S \bowtie^b E = \langle 2n_1, \dots, 2n_r, 2m_1 + b, \dots, 2m_s + b \rangle.$$

In particular, we recall that $K(S)$ is minimally generated by the elements $f(S) - x$, where $x \in \text{PF}(S)$, and therefore, if $E = K(S) + z$, the semigroup $S \bowtie^b E$ is generated by

$$\{2n_i, 2(f(S) - x_j + z) + b \mid n_i \in G(S), x_j \in \text{PF}(S)\};$$

moreover if S is almost symmetric, it follows from Theorem 2.2 that $S \bowtie^b (K(S) + z)$ is minimally generated by $\{2n_i, 2z + b, 2x_j + 2z + b \mid n_i \in G(S), x_j \in \text{PF}(S) \setminus \{f(S)\}\}$. Finally we remember that if $S = \langle s_1, \dots, s_\nu \rangle$ is a symmetric numerical semigroup then $k[[S]] := k[[t^{s_1}, \dots, t^{s_\nu}]]$ is a one-dimensional Gorenstein local ring for any field k .

For each $h \geq 4$, $h \notin \{14 + 22k, 35 + 46k \mid k \in \mathbb{N}\}$, Construction 2.6 allows to produce Gorenstein rings whose Hilbert function decreases at level h , while for $h = 3$ we can use Example 2.10.ii. The next example is useful to complete the case $h = 2$.

Example 3.1. Consider the numerical semigroup

$$S = \langle 68, 72, 78, 82, 107, 111, 117, 121, 158, 162, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, \\ 186, 188, 190, 192, 194, 196, 197, 198, 200, 201, 202, 205, 206, 207, 209, 210, 211, 213, 215, \\ 217, 219, 221, 223, 225, 227, 229, 231, 233, 235, 237, 239, 241, 245, 249 \rangle.$$

It is almost symmetric, has type 53, and its Hilbert function is $[1, 54, 52, 50, 54, 64, 68 \rightarrow]$. Consequently by Proposition 1.4, $k[[S \bowtie^b K]]$ is a Gorenstein ring and has Hilbert function $[1, 107, 106, 102, 104, 118, 132, 136 \rightarrow]$ decreasing at level 2 for any canonical ideal K and for any odd $b \in S$.

The next lemma will allow us to show that in general, even if R is Gorenstein, there are no bounds for $H_R(h-1) - H_R(h)$.

Lemma 3.2. *Let $R^{(0)}$ be a local ring. The following hold:*

- i. *Consider the ring $R^{(i+1)} := R^{(i)}(\mathfrak{m}^{(i)})_{a^{(i)}, b^{(i)}}$, where $\mathfrak{m}^{(i)}$ is the maximal ideal of $R^{(i)}$ and $a^{(i)}, b^{(i)}$ are two elements of $R^{(i)}$. Then*

$$H_{R^{(i)}}(h) = 2H_{R^{(i-1)}}(h) = \dots = 2^i H_{R^{(0)}}(h) \text{ for any } h > 0.$$

ii. If $R^{(0)}$ is almost Gorenstein and has Cohen-Macaulay type t , then $R^{(i)}$ is almost Gorenstein and, if R^0 is not a DVR, it has Cohen-Macaulay type $2^i t + 2^i - 1$.

Proof. The first point is a straightforward application of Proposition 1.1, while the second one follows from [7, Proposition 2.9], which says that R is almost Gorenstein if and only if $R(\mathfrak{m})_{a,b}$ is almost Gorenstein and in this case, if R is not a DVR, the Cohen-Macaulay type of $R(\mathfrak{m})_{a,b}$ is $2t + 1$. \square

Theorem 3.3. *For any integers $m \geq 1$ and $h > 1$, $h \notin \{35 + 46k, 14 + 22k \mid k \in \mathbb{N}\}$, there exist infinitely many non-isomorphic one-dimensional Gorenstein local rings R such that $H_R(h - 1) - H_R(h) > m$.*

Proof. If $h = 2$ consider the ring $R^{(0)} = k[[S]]$ of Example 3.1, then $R^{(i)}$ is almost Gorenstein and $H_{R^{(i)}}(1) = 54 \cdot 2^i$, $H_{R^{(i)}}(2) = 52 \cdot 2^i$, $t(R^{(i)}) = 54 \cdot 2^i - 1$ by the previous lemma. We achieve the proof using Corollary 1.5 applied with $R = R^{(i)}$: $H_{R(I)_{a,b}}(1) - H_{R(I)_{a,b}}(2) = t(R) - H_R(2) = 54 \cdot 2^i - 1 - 52 \cdot 2^i = 2^{i+1} - 1 > m$, if $i \geq i_0 = \lfloor \log_2(m + 1) \rfloor$.

If $h \geq 3$, consider an almost Gorenstein ring R such that $H_R(h - 2) - H_R(h) = n > 0$, whose existence we proved in Theorem 2.9 for $h \geq 4$ and in Example 2.10.ii for $h = 3$; then apply the construction of the previous lemma with $i_0 = \lfloor \log_2(m/n) \rfloor + 1$. With the notation of the previous lemma, it follows that $H_{R^{(i)}}(h - 2) - H_{R^{(i)}}(h) = 2^i n > m$ for any $i \geq i_0$.

Now, for each h of the statement, if $i \geq i_0$, ω is a canonical ideal of $R^{(i)}$ and $a, b \in R^{(i)}$, Corollary 1.5 implies that the ring $R^{(i)}(\omega)_{a,b}$ has all the properties we are looking for.

Clearly, for any $i \geq i_0$ we get infinitely many non-isomorphic rings, because their Hilbert functions are different. \square

From the proof it is clear that, for any such m and h , in the rings of the previous theorem there are always non-reduced rings (idealization), reduced rings that are not integral domains (amalgamated duplication), and numerical semigroup rings (numerical duplication).

Example 3.4. Consider the first numerical semigroup S of Example 2.8. Set $b = 33$ and $E = K(S) + 101 = K(S) + f(S) + 1 \subseteq S$. Since we know the generators and the pseudo-Frobenius numbers of S , it follows from above that $k[[S \bowtie^b E]]$ is equal to

$$k[[t^{64}, t^{66}, t^{76}, t^{138}, t^{144}, t^{146}, t^{148}, t^{150}, t^{154}, t^{156}, t^{158}, t^{160}, t^{162}, t^{164}, t^{166}, t^{168}, t^{170}, t^{172}, t^{174}, t^{176}, \\ t^{178}, t^{180}, t^{182}, t^{184}, t^{186}, t^{188}, t^{190}, t^{235}, t^{309}, t^{313}, t^{315}, t^{317}, t^{319}, t^{321}, t^{323}, t^{325}, t^{327}, t^{329}, t^{331}, t^{333}, \\ t^{335}, t^{337}, t^{339}, t^{341}, t^{343}, t^{345}, t^{347}, t^{349}, t^{351}, t^{353}, t^{355}, t^{357}, t^{361}]]$$

and is a one-dimensional Gorenstein local ring. Moreover Proposition 1.4 implies that its Hilbert function is $[1, 53, 54, 54, 53, 53, 56, 59, 61, 63, 64 \rightarrow]$.

In the next example we show how the construction of Lemma 3.2 and Theorem 3.3 works.

Example 3.5. Let $T^{(0)}$ be the second semigroup of Example 2.8 and construct the numerical semigroups of Lemma 3.2 applying the numerical duplication, that can be considered a particular case of the lemma.

- $T^{(0)}$ is almost symmetric with type 37 and

$$H_{T^{(0)}} = [1, 38, 38, 38, 38, 37, 44, \rightarrow];$$

- $T^{(1)} := T^{(0)} \bowtie^{53} M(T^{(0)})$ is almost symmetric with type 75 and

$$H_{T^{(1)}} = [1, 76, 76, 76, 76, 74, 88 \rightarrow];$$

- $T^{(2)} := T^{(1)} \bowtie^{141} M(T^{(1)})$ is almost symmetric with type 151 and

$$H_{T^{(2)}} = [1, 152, 152, 152, 152, 148, 176 \rightarrow];$$

- $T^{(3)} := T^{(2)} \bowtie^{317} M(T^{(2)})$ is almost symmetric with type 303 and

$$H_{T^{(3)}} = [1, 304, 304, 304, 304, 296, 352 \rightarrow];$$

- $T^{(4)} := T^{(3)} \bowtie^{669} M(T^{(3)})$ is almost symmetric with type 607 and

$$H_{T^{(4)}} = [1, 608, 608, 608, 608, 592, 704 \rightarrow];$$

- $T := T^{(4)} \bowtie^{1373} K$, where $K := K(T^{(4)}) + f(T^{(4)}) + 1 \subseteq T^{(4)}$, is symmetric and has Hilbert function

$$H_T = [1, 1215, 1216, 1216, 1216, 1200, 1296, 1408 \rightarrow].$$

If we are looking for symmetric semigroups with bigger difference between $H(4)$ and $H(5)$, we can continue in this way before to consider the numerical duplication with respect to a canonical ideal. Anyway, we note that in this example T has 1215 minimal generators included between 1408 and 23835.

Example 3.6. Consider the almost symmetric numerical semigroup

$$T_0 = \langle 30, 33, 37, 64, 68, 69, 71, 72, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92 \rangle$$

that has Hilbert function $[1, 26, 26, 25, 24, 27, 28, 29, 30 \rightarrow]$. Let $K'(T_i)$ be a proper canonical ideal of T_i and b_i an arbitrary odd element of T_i . All the following numerical semigroups are symmetric:

- The semigroup $T_1 := T_0 \bowtie^{b_0} K'(T_0)$ has Hilbert function

$$H_{T_1} = [1, 51, 52, 51, 49, 51, 55, 57, 59, 60 \rightarrow];$$

- The semigroup $T_2 := T_1 \bowtie^{b_1} K'(T_1)$ has Hilbert function

$$H_{T_2} = [1, 52, 103, 103, 100, 100, 106, 112, 116, 119, 120 \rightarrow];$$

- The semigroup $T_3 := T_2 \bowtie^{b_2} K'(T_2)$ has Hilbert function

$$H_{T_3} = [1, 53, 155, 206, 203, 200, 206, 218, 228, 235, 239, 240 \rightarrow];$$

- The semigroup $T_4 := T_3 \bowtie^{b_3} K'(T_3)$ has Hilbert function

$$H_{T_4} = [1, 54, 208, 361, 409, 403, 406, 424, 446, 463, 474, 479, 480 \rightarrow];$$

- The semigroup $T_5 := T_4 \bowtie^{b_4} K'(T_4)$ has Hilbert function

$$H_{T_5} = [1, 55, 262, 569, 770, 812, 809, 830, 870, 909, 937, 953, 959, 960 \rightarrow].$$

Since a symmetric numerical semigroup is almost symmetric the Hilbert functions above can be computed from the one of T_0 by means of Proposition 1.4.

The next two examples show that it is possible to find symmetric semigroups with decreasing Hilbert function even if we start with *non-almost symmetric semigroups*.

Further we recall that, by [21, Corollary 4.11], in a symmetric semigroup with decreasing Hilbert function the difference between the multiplicity and the embedding dimension has to be greater or equal to 5: in the following example is 6.

Example 3.7. Consider $S := \langle 30, 33, 37, 64, 68, 69, 71, 72, 73, 75, \longrightarrow 89, 91, 92, 95 \rangle$ that has Hilbert function $[1, 27, 26, 25, 24, 27, 28, 29, 30, \rightarrow]$ and set $K := K(S) + 66 \subseteq S$. Then the semigroup $S \bowtie^{33} K$ is symmetric and has Hilbert function $[1, 54, 55, 55, 54, 57, 58, 59, 60 \rightarrow]$. We also note that S is not almost symmetric by Proposition 1.4.

It is possible to define the numerical duplication $S \bowtie^b E$, even if the ideal E is *not contained* in S ; in this case we have to require that $E + E + b \subseteq S$, that is true if $E \subseteq S$, otherwise the set $S \bowtie^b E$ is not a numerical semigroup. In [29, Corollary 3.10] it is proved that, even if E is not proper, $S \bowtie^b E$ is symmetric if and only if E is a canonical ideal; actually every symmetric numerical semigroup can be constructed as $S \bowtie^b K(S)$ for some S and some odd $b \in S$ (see also [29, Proposition 3.3] and [30, Section 3]). However, if *the ideal is not proper*, the Hilbert function of the numerical duplication can be different from the expected one; on the other hand the next examples show that also in this case it is possible to find symmetric semigroups with decreasing Hilbert function.

Example 3.8. In [21, Example 3.7] it is showed the following numerical semigroup

$$S = \langle 30, 33, 37, 73, 76, 77, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92, 94, 95, 98, 101, 108 \rangle$$

that has Hilbert function $[1, 24, 25, 24, 23, 25, 27, 29, 30 \rightarrow]$.

Let $K := K(S)$. The following semigroups are all symmetric and all of them, but H_2 , have decreasing Hilbert function. It is easy to see that, for the following choices of b , one has $K + K + b \subseteq S$, but $S \bowtie^b K$ cannot be realized as a numerical duplication with respect to a proper ideal.

- The semigroup $H_1 := S \bowtie^{79} K$ has Hilbert function $[1, 44, 41, 40, 52, 58, 60, \rightarrow]$;
- The semigroup $H_2 := S \bowtie^{81} K$ has Hilbert function $[1, 43, 45, 47, 52, 54, 56, 58, 60 \rightarrow]$;
- The semigroup $H_3 := S \bowtie^{85} K$ has Hilbert function $[1, 44, 42, 45, 52, 54, 58, 60 \rightarrow]$;
- The semigroup $H_4 := S \bowtie^{87} K$ has Hilbert function $[1, 46, 48, 47, 49, 51, 56, 58, 60 \rightarrow]$;
- The semigroup $H_5 := S \bowtie^{93} K$ has Hilbert function $[1, 47, 49, 48, 48, 50, 55, 58, 60 \rightarrow]$.

Note that H_5 have the same Hilbert function of the numerical duplication with respect to a proper canonical ideal of S .

If one consider the semigroups constructed in the previous section and their numerical duplications with respect to non proper canonical ideals, it is possible to find symmetric semigroups whose Hilbert functions *decrease at more levels*. For instance the next example shows a symmetric semigroup that decreases 13 times. We also note that it decreases at level 14, thus this suggests that the restrictions of Theorem 3.3 can be removed.

Example 3.9. Let S be the semigroup of Construction 2.6 with $\ell = 15$, that has 258 minimal generators. According to GAP [17], the symmetric semigroup $S \bowtie^{957} K(S)$ has Hilbert function

$$[1, 514, 514, 513, 512, 511, 510, 509, 508, 507, 506, 505, 504, 503, 502, 500, 523, H_S(17), \dots].$$

In the last example we show that the numerical duplication of S with respect to a canonical ideal can have decreasing Hilbert function, even if that of S is non-decreasing.

Further, among the symmetric semigroups with decreasing Hilbert function this is the semigroup with *the smallest multiplicity and embedding dimension* that we know.

Example 3.10. Consider $S = \langle 19, 21, 24, 47, 49, 50, 51, 52, 53, 54, 55, 56, 58, 60 \rangle$, shown in [21, Example 3.2.1], and let $T = S \bowtie^{49} K(S)$. The Gorenstein local ring $k[[T]]$ is

$$k[[t^{38}, t^{42}, t^{48}, t^{49}, t^{94}, t^{100}, t^{101}, t^{102}, t^{104}, t^{105}, t^{106}, t^{107}, t^{108}, t^{109}, t^{110}, t^{111}, t^{112}, t^{113}, t^{115}, t^{116}, t^{117}, t^{119}, t^{120}, t^{121}, t^{123}, t^{127}]].$$

Even if $k[[S]]$ has non-decreasing Hilbert function $[1, 14, 14, 14, 16, 18, 19 \rightarrow]$, the Hilbert function of $k[[T]]$ is $[1, 26, 25, 25, 32, 38 \rightarrow]$; we also note that its multiplicity is 38.

4. Appendix

In this appendix we illustrate the technical lemmas necessary to prove Theorem 2.9; in the sequel we shall assume e, n_1, n_2, ℓ, S be as defined in Construction 2.6.

Lemma 4.1. *We have:*

- (i) *If ℓ is odd then $GCD(e, n_1, n_2) = 1 \iff \ell \notin \{35 + 46k, k \in \mathbb{N}\}$.*
- (ii) *If ℓ is even then $GCD(e, n_1, n_2) = 1 \iff \ell \notin \{14 + 22k, k \in \mathbb{N}\}$.*

Proof. i. $\begin{cases} 2\ell - 1 = ab \\ \ell^2 - 6 = ac, \ a, b, c \in \mathbb{N} \end{cases} \iff \begin{cases} 2\ell = ab + 1 \\ 4\ell^2 - 24 = 4ac = a^2b^2 + 2ab - 23 \end{cases}$

$\implies a(ab^2 + 2b - 4c) = 23$, hence if $a > 1 \implies \begin{cases} a = 23 \\ (ab^2 + 2b - 4c) = 1 \end{cases} \implies$

$23b^2 + 2b - 4c = 1$, and so $b = 2q + 1$, for some $q \in \mathbb{N}$. Therefore the last equation becomes $92q^2 + 96q + 24 = 4c$, that is $c = 23q^2 + 24q + 6$.

Hence $\begin{cases} \ell = 23q + 12 \\ b = 2q + 1 \\ c = 23q^2 + 24q + 6. \end{cases} \ell \text{ odd} \implies q = 2k + 1, k \in \mathbb{N} \implies \begin{cases} \ell = 46k + 35 \\ a = 23, \ b = 2q + 1 \\ c = 23q^2 + 24q + 6. \end{cases}$

Further, since $e = (\ell + 2)(2\ell - 1) - (\ell^2 - 6)$, we see that if $\ell \in \{35 + 46k, k \in \mathbb{N}\}$, then $GCD(e, n_1, n_2) \neq 1$.

ii. Since $(\ell^2 - \ell - 6) = (\ell - 3)(\ell + 2)$, as above we get

$$\begin{cases} \ell - 3 = ab \\ \ell^2 + 3\ell + 4 = ac, \ a, b \text{ odd}, c \text{ even} \end{cases} \iff \begin{cases} \ell = ab + 3 \\ a(ab^2 + 9b - c) = -22 \end{cases}$$

hence $a = 11, \ell = 11b + 3 = 14 + 22k, k \in \mathbb{N}$. □

Lemma 4.2. *Let $\Gamma' := \{kn_1, k \in [1, \ell]\} \cup \{n_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\} \cup \{t_1, t_2\}$. Then:*

- (i) $\ell n_1 - n_2 = (\ell + 1)n_2 - (\ell - 1)e = s_{0, \ell+1}$
 $\ell n_1 + e - s_{0, q'} \in \{s_{0, q'}, 2 \leq q' \leq \ell\}$, for each $2 \leq q \leq \ell$.
- (ii) $e < n_1 < n_2$ are the lowest elements in Γ' and ℓn_1 is the greatest element in Γ' .

Proof. i. The equality follows from $\ell n_1 = (\ell+2)n_2 - (\ell-1)e$, see Construction 2.6. Moreover: $\ell n_1 + e - s_{0,q} = \ell n_1 + e - (qn_2 - (q-2)e) = (\ell+2-q)n_2 - (\ell-q)e = q'n_2 - (q'-2)e$, with $q' = \ell+2-q \in [2, \ell]$.

ii. The first statement follows by a direct check. To see that ℓn_1 is the greatest element, first note that $\ell n_1 > kn_1$, if $1 \leq k < \ell$, and $\ell n_1 > n_2$. Moreover:

$$\ell n_1 - t_1 = -n_1 + (\ell-1)e = -F + (\ell-2)e > 2e - F > 0.$$

$$\ell n_1 - t_2 = (\ell+1)n_1 - \ell e > 0.$$

Now let ℓ be odd and consider $d = \ell n_1 - n$, where $n = 2e + aF + bG \in \{s_{p,q}\} \cup \{r_{p',q'}\}$ with $-\ell \leq a \leq \ell$ and $1 \leq b \leq \ell+1$:

$$d = (\ell-a)F - bG + (\ell-2)e = (\ell-2)e + (\ell-a-b)F - b(G-F) \geq (\ell-2)e - F - (\ell+1)(\ell^2 - 2\ell - 5) = (\ell-2)(\ell^2 + 3\ell + 4) - (2\ell-1) - (\ell+1)(\ell^2 - 2\ell - 5) = 2\ell^2 + 3\ell - 2 = n_2 > 0.$$

If ℓ is even, then $F = \ell-3$, $G = \ell^2 - \ell - 6$ and we get

$$d = (\ell-2)e + (\ell-a-b)F - b(G-F) \geq (\ell-2)(\ell^2 + 3\ell + 4) - (\ell-3) - (\ell+1)(\ell^2 - 2\ell - 3) = n_2 > 0.$$

□

Lemma 4.3. Denote respectively by $F = (2\ell-1)$, $G = (\ell^2-6)$, if ℓ is odd, and by $F = (\ell-3)$, $G = (\ell^2 - \ell - 6)$, if ℓ is even. Then :

- (i) $n \in \{s_{p,q}\} \cup \{r_{p,q}\} \implies n = 2e + aF + bG$ with $a+b \in [1, \ell+1]$, $a \in [-\ell, \ell]$, and $b \in [1, \ell+1]$.
- (ii) If $n, n' \in \{s_{p,q}\} \cup \{r_{p,q}\}$, then $n - n' = aF + bG$ with $-\ell \leq a+b, b \leq \ell$ and $-2\ell \leq a \leq 2\ell$.
- (iii) If $aF + bG = he$, with $a, a+b \in [-2\ell-3, 2\ell+3]$, $b \in [-\ell-1, 2\ell+2]$ then a, b, h verify one of the following systems:

$$(1) \begin{cases} a = \nu(\ell+2) \\ b = -\nu \\ a+b = \nu(\ell+1) \\ -1 \leq \nu \leq 1 \\ h = \nu, \text{ if } \ell \text{ odd} \\ h = 0, \text{ if } \ell \text{ even} \end{cases} \quad (2) \begin{cases} a = 2 \\ b = \ell+1 \\ a+b = \ell+3 \\ h = \ell-2, \text{ if } \ell \text{ odd} \\ h = \ell-3, \text{ if } \ell \text{ even} \end{cases} \quad (3) \begin{cases} a = -\ell \\ b = \ell+2 \\ a+b = 2 \\ h = \ell-3 \end{cases} \quad (4) \begin{cases} a = -2\ell-2 \\ b = \ell+3 \\ a+b = 1-\ell \\ h = \ell-4, \text{ if } \ell \text{ odd} \\ h = \ell-3, \text{ if } \ell \text{ even} \end{cases}$$

where systems (2), (3), (4), describe the possible situations with $h > 0$; if $h < 0$, then besides case (1) one has case (2bis) obtained from (2) by changing a, b, h in their opposite numbers.

Proof. i. Note that $e = -\ell + (\ell+2)^2$. Therefore if ℓ is odd:

$$\begin{cases} e &= (\ell+2)(2\ell-1) - (\ell^2-6) = (\ell+2)F - G \\ (\ell-3)e &= -\ell(2\ell-1) + (\ell^2-6)(\ell+2) \end{cases}$$

$$\begin{cases} s_{p,q} &= pn_1 + qn_2 - (p+q-2)e &= 2e + (\ell-1)p + (\ell^2-6)q \\ r_{p,q} &= \ell n_1 + e - s_{p,q} &= 2e - (\ell-1)p + (\ell^2-6)(\ell+2-q) \\ & &= 2e + (\ell-1)p' + (\ell^2-6)q' \end{cases}$$

while if ℓ is even we have $(\ell-3)e = -\ell(\ell-3) + (\ell^2 - \ell - 6)(\ell+2)$ and

$$\begin{cases} s_{p,q} &= pn_1 + qn_2 - (p+q-2)e &= 2e + (\ell-3)p + (\ell^2 - \ell - 6)q \\ r_{p,q} &= \ell n_1 + e - s_{p,q} &= 2e - (\ell-3)p + (\ell^2 - \ell - 6)(\ell+2-q) \\ & &= 2e + (\ell-3)p' + (\ell^2 - \ell - 6)q' \end{cases}$$

where
$$\begin{cases} 2 \leq p+q \leq \ell+1, & 0 \leq p \leq \ell, & 1 \leq q \leq \ell+1 \\ 1 \leq p'+q' \leq \ell, & -\ell \leq p' \leq 0, & 1 \leq q' \leq \ell+1 \end{cases}$$

ii. It is immediate by part i.

iii. Let ℓ be *odd* and $a(2\ell-1) + b(\ell^2-6) = he = h[(\ell+2)(2\ell-1) - (\ell^2-6)]$. Then $[a - h(\ell+2) + \mu(\ell^2-6)](2\ell-1) + [b + h - \mu(2\ell-1)](\ell^2-6) = 0$, $\forall \mu \in \mathbb{Z}$, therefore

$$\begin{cases} a - h(\ell+2) = -\mu(\ell^2-6) \\ b + h = \mu(2\ell-1), \end{cases} \implies \begin{cases} h = \mu(2\ell-1) - b \\ a - [\mu(2\ell-1) - b](\ell+2) = -\mu(\ell^2-6) \end{cases} \implies a + b(\ell+2) = \mu e.$$

If ℓ is *even*, let $a(\ell-3) + b(\ell^2-\ell-6) = a(\ell-3) + b(\ell-3)(\ell+2) = (\ell-3)[a + b(\ell+2)] = he$. Since $\ell-3$ divides the first member and $(\ell-3, e) = 1$ by the previous lemma, it follows that $h = (\ell-3)\mu$ for some μ and then $a + b(\ell+2) = \mu e$. Hence in both cases we obtain the equality $a + b(\ell+2) = \mu e$.

Now consider $a + b(\ell+2) = \mu e$; since $-\ell \cdot 1 + (\ell+2)(\ell+2) = e$, we get:
$$\begin{cases} a = -\mu\ell + \nu(\ell+2) \\ b = \mu(\ell+2) - \nu \\ a + b = 2\mu + \nu(\ell+1) \end{cases}$$

By the assumptions $a, a+b \in [-2\ell-3, 2\ell+3]$, we can assume $h, \mu \geq 0$ (the cases with $h < 0$ can be obtained by changing the values of $a, b, a+b$ found for $h > 0$, with $b \leq \ell+1$ in their opposite). Hence:

$$\mu = 0, a, a+b \in [-2\ell-3, 2\ell+3] \implies (1) \begin{cases} a = \nu(\ell+2), \\ b = -\nu, \\ a + b = \nu(\ell+1) \text{ with } \nu \in [-1, 1] \\ h = \nu, \text{ if } \ell \text{ odd}, \\ h = 0, \text{ if } \ell \text{ even} \end{cases}$$

$\mu > 0 \implies -1 \leq \nu \leq 1$; in fact $a + b \in [-2\ell-3, 2\ell+3] \implies \nu \leq 1$, and $-\ell + \nu(\ell+2) \geq a = -\ell + \nu(\ell+2) - (\mu-1)\ell \geq -2\ell-3 \implies \nu \geq -1$ ($\nu = -1 \implies \mu = 1$).

Moreover if $\nu = 0, 1$, then $b = \mu(\ell+2) - \nu = \mu(\ell+1) + \mu - \nu \leq 2\ell+2$ implies $\mu \leq 1$. Hence $\nu \in \{-1, 0, 1\}$, $\mu = 1$ and we get the systems (2), (3), (4) of the thesis. \square

Lemma 4.4. *As above, let $\Gamma' := \{kn_1 \mid k \in [1, \ell]\} \cup \{n_2\} \cup \{s_{p,q}, r_{p',q'}\} \cup \{t_1, t_2\}$. Then:*

- (i) *The elements of Γ' are all non-zero (mod e);*
- (ii) *The elements of Γ' have distinct residues (mod e);*
- (iii) *If $m, n, n' \in \Gamma'$, the equality $m + n = n' + \alpha e$ implies either $\alpha > 0$ or $\alpha = 0$ and $n' \in \Gamma'' := \{n_1 + n_2, 2n_2, kn_1 \mid 2 \leq k \leq \ell\}$;*

Proof. i. In fact for any element $s \in \Gamma'$, we have $s = aF + bG + ke$. Then if $s = \lambda e$ for some $\lambda \geq 0$, it follows that $aF + bG = he$ with $h = \lambda - k$. Note that

$$t_1 = (\ell+1)F + 2e, \quad t_2 = -F + (\ell-1)e.$$

Then:

$$\left[\begin{array}{lll} s \in \{s_{p,q}\} \cup \{r_{p',q'}\} & aF + bG = (\lambda - 2)e & a + b \in [1, \ell + 1] \quad b \in [1, \ell + 1], a \in [-\ell, \ell] \\ s = t_1 & (\ell + 1)F = (\lambda - 2)e & a + b = a = \ell + 1 \quad b = 0 \\ s = t_2 & F = (\ell - \lambda - 1)e & a + b = a = 1 \quad b = 0 \\ s = kn_1 & kF = (\lambda - k)e & a + b = a = k \leq \ell \quad b = 0 \end{array} \right.$$

Every case verifies the assumptions of Lemma 4.3.iii. Hence we can apply this result: note that in cases (1) ... (4) of Lemma 4.3.iii, either $b = -\nu \in [-1, 1]$ or $b \geq \ell + 1$, and $b = \ell + 1 \iff a + b = \ell + 3$, $b = -\nu \iff a + b = \nu(\ell + 1)$. Easily we can see that no case is possible.

ii. Let $m, n \in \Gamma'$ and let $m = n + he, h > 0$. Then $m = pF + qG + ke, n = p'F + q'G + k'e \implies he = m - n = (p - p')F + (q - q')G + (k - k')e \implies aF + bG = (h - k + k')e$: by recalling the above table and Lemma 4.3.i,ii. we list the possible $c_i = m - n$:

$$\left[\begin{array}{lll} c_1 : m, n \in \{s_{p,q}, r_{p',q'}\} & a + b & b \\ c_2 : m \in \{s_{p,q}, r_{p',q'}\}, n = kn_1 & [-\ell, \ell], & [-\ell, \ell] \\ c_3 : m \in \{s_{p,q}, r_{p',q'}\}, n = n_2 & [1 - k, \ell - k + 1] & [1, \ell + 1] \\ c_4 : m \in \{s_{p,q}, r_{p',q'}\}, n = t_1 & [0, \ell] & [0, \ell] \\ c_5 : m \in \{s_{p,q}, r_{p',q'}\}, n = t_2 & [-\ell, 0] & [1, \ell + 1] \\ c_6 : m \in \{s_{p,q}, r_{p',q'}\}, n = t_2 & [2, \ell + 2] & [1, \ell + 1] \\ c_7 : m = kn_1, n = n_2 & k - 1 & -1 \\ c_8 : m = kn_1, n = t_1 & k - \ell - 1 & 0 \\ c_9 : m = kn_1, n = t_2 & k + 1 & 0 \\ c_{10} : m = t_1, n = n_2 & \ell & -1 \\ c_{11} : m = t_1, n = t_2 & \ell + 2 & 0 \\ c_{12} : m = t_2, n = n_2 & -2 & -1 \end{array} \right.$$

As in i, by Lemma 4.3.iii one can easily see that no case is possible.

iii. We denote by:

$$p_1 = \begin{cases} kn_1 = kF + ke & a + b = a = k \leq \ell \quad b = 0, \text{ or} \\ t_1 = (\ell + 1)F + 2e & a + b = a = \ell + 1 \quad b = 0 \text{ or} \\ t_2 = -F + (\ell - 1)e & a + b = a = -1 \quad b = 0 \end{cases}$$

We can write $p_1 = aF + 0G + \delta e$ $\delta \in [1, \ell]$, $a + b = a \in [-1, \ell + 1] \setminus \{0\}$, $b = 0$.

We divide the elements of Γ' in three types:

$$\left[\begin{array}{lll} p_1 = aF + \delta e & \delta \in [1, \ell], & a + b \in [-1, \ell + 1] \quad b = 0 \\ n_2 = G + e & & a + b = 1 \quad b = 1, a = 0 \\ p_3 = aF + bG + 2e \in \{s_{p,q}, r_{p',q'}\} & a + b \in [1, \ell + 1] & b \in [1, \ell + 1], a \in [-\ell, \ell] \end{array} \right.$$

Denote by σ_1 any sum $p_1 + p'_1$:

$$\sigma_1 = \left[\begin{array}{lll} kn_1 + hn_1 = (k + h)F + (k + h)e & a + b \in [2, 2\ell] & b = 0 \\ kn_1 + t_1 = (\ell + 1 + k)F + (2 + k)e & a + b = a = (\ell + 1 + k) & b = 0 \\ kn_1 + t_2 = (k - 1)F + (k + \ell - 1)e & a + b = a = k - 1 & b = 0 \\ 2t_1 = (2\ell + 2)F + 4e & a + b = 2\ell + 2 & b = 0 \\ t_1 + t_2 = \ell F + (\ell + 1)e & a + b = \ell & b = 0 \\ 2t_2 = -2F + (2\ell - 2)e & a + b = -2 & b = 0 \end{array} \right.$$

Denote by σ_2 the sum $p_1 + n_2$:

$$\sigma_2 = \begin{cases} kn_1 + n_2 &= kF + 1G + (k+1)e & a+b = k+1 & a=k & b=1 \\ t_1 + n_2 &= (\ell+1)F + 1G + 3e & a+b = \ell+2 & a=\ell+1 & b=1 \\ t_2 + n_2 &= -F + 1G + \ell e & a+b = 0 & a=-1 & b=1 \end{cases}$$

Let $p_3, p'_3 \in \{s_{p,q}, r_{p,q}\}$, where $p_3 = a'F + b'G + 2e$, $p'_3 = a''F + b''G + 2e$, and denote by $\sigma_3 = aF + bG + \beta e$ any sum $p_3 + p_1, p_3 + n_2, p_3 + p'_3$:

$$\sigma_3 = \begin{cases} p_3 + kn_1 &= (k+a')F + b'G + (k+2)e & a+b \in [k+1, \ell+k+1] & b \in [1, \ell+1] \\ p_3 + t_1 &= (\ell+1+a')F + b'G + 4e & a+b \in [\ell+2, 2\ell+2] & b \in [1, \ell+1] \\ p_3 + t_2 &= (a'-1)F + b'G + (\ell+1)e & a+b \in [0, \ell] & b \in [1, \ell+1] \\ p_3 + n_2 &= a'F + (b'+1)G + 3e & a+b \in [2, \ell+2] & b \in [2, \ell+2] \\ p_3 + p'_3 &= (a'+a'')F + (b'+b'')G + 4e & a+b \in [2, 2\ell+2] & b \in [2, 2\ell+2] \end{cases}$$

In conclusion we can write:

$$\begin{cases} \sigma_1 &= aF + \lambda e, & \lambda \in [2, 2\ell] & a = a+b \in [-2, 2\ell+2] & b=0 \\ \sigma_2 &= aF + G + \mu e, & \mu \in [2, \ell+1] & a+1 = a+b \in [0, \ell+2] & b=1 \\ 2n_2 &= 2G + 2e, & \mu=2 & a=0 & b=2 \\ \sigma_3 &= aF + bG + \nu e, & \nu \in [3, \ell+2] & a \in [-2\ell, 2\ell] & a+b \in [0, 2\ell+2] & b \in [1, 2\ell+2] \end{cases}$$

$$\begin{cases} p_1 &= aF + \delta e & \delta \in [1, \ell], & a+b \in [-1, \ell+1] = a & b=0 \\ n_2 &= G + e & & a+b=1 & a=0 & b=1 \\ p_3 &= aF + bG + 2e \in \{s_{p,q}, r_{p,q}\} & & a+b \in [1, \ell+1] & a \in [-\ell, \ell] & b \in [1, \ell+1] \end{cases}$$

Further note that, $\sigma_i = n_2 + \alpha e \implies \alpha > 0$ by Lemma 4.2.ii, since $2n_1 > n_2$.

Let $m+n = \sigma_i$, $n' = p_j$, $1 \leq i, j \leq 3$, assume $\sigma_i = p_j + \alpha e$ and consider the following table:

$$\begin{cases} \sigma_1 - p_1 = aF + ue & u \in [2-\ell, 2\ell-1] & a+b \in [-\ell-3, 2\ell+3] & b=0 \\ \sigma_1 - p_3 = aF + bG + ue, & u \in [0, 2\ell-2] & a+b \in [-\ell-3, 2\ell+1] & b \in [-\ell-1, -1] \\ \sigma_2 - p_1 = aF + 1G + ue, & u \in [2-\ell, \ell] & a+b \in [-\ell-1, \ell+3] & b=1 \\ \sigma_2 - p_3 = aF + bG + ue, & u \in [0, \ell-1] & a+b \in [-\ell-1, \ell+1] & b \in [-\ell, 0] \\ 2n_2 - p_1 = aF + 2G + ue, & u \in [2-\ell, 1] & a+b \in [-\ell+1, 3] & b=2 \\ 2n_2 - p_3 = aF + bG + 0e, & u=0 & a+b \in [-\ell+1, 1] & b \in [-\ell+1, 1] \\ \sigma_3 - p_1 = aF + bG + ue, & u \in [3-\ell, \ell+1] & a+b \in [-\ell-1, 2\ell+3] & b \in [1, 2\ell+2] \\ \sigma_3 - p_3 = aF + bG + ue, & u \in [1, \ell] & a+b \in [-\ell-1, 2\ell+1] & b \in [-\ell, 2\ell+1] \end{cases}$$

$$\sigma_i - p_j = aF + bG + ue = \alpha e \iff aF + bG = he, \quad h = \alpha - u \quad (\alpha = h + u) \quad (*)$$

To prove that, in all possible cases, either $\alpha > 0$ or $\alpha = 0$ and $p_j \in \Gamma''$, we can apply Lemma 4.3.iii, since the integers $a, b, a+b$ verify the required assumptions. It is straightforward to see that the cases $2n_2 - p_1$ and $2n_2 - p_3$ are impossible, except when $2n_2 = p_3$.

- Case $2n_2 = p_3$: this equality means $2G + 2e = a'F + b'G + 2e \implies a'F + (b'-2)G = 0$, with $a' \in [-\ell, \ell]$, $b' \in [1, \ell+1]$. Hence we are in case (1) of Lemma 4.3.iii, with $h = 0$ and so either $b'-2 = 0, a = 0, p_3 = 2n_2$, or ℓ even, $b'-2 = \pm 1, a = \pm(\ell+2)$, that are impossible.
- Case $\sigma_1 - p_1$: we have $b = 0$ and then $a+b = a = h = 0$, by Lemma 4.3.iii. If $p_1 \notin \Gamma''$, it is easy to see that $p_1 = t_1$ and thus $\sigma_1 = (\ell+1)F + (\ell+1)e$, because $a = 0$. In this case $\alpha = u = \ell+1-2 > 0$.
- Case $\sigma_1 - p_3$: we have $u \geq 0$ and $b \in [-\ell-1, -1]$ thus $h \geq 0$ and α is always positive,

except when either $h = u = 0$, $b = -1, a = \ell + 2$, ℓ even, or $a = -2, b = -\ell - 1, h \in [2 - \ell, 3 - \ell]$. In the first case, it is easy to see that $u = h = 0$ implies $\sigma_1 = 2n_1$. From these cases follows that $a + b \geq \ell + 1$, but it is impossible when $\sigma_1 = 2n_1$. In the second one, $p_3 = (\ell + 1)G + 2e \implies \sigma_1 = 2t_2 = -2F + (2\ell - 2)e \implies u = 2\ell - 4, \alpha \geq \ell - 2 > 0$.

- Case $\sigma_2 - p_1$: we have $b = 1, a + b = -\ell - 1$, and $h \in \{-1, 0\}$ by Lemma 4.3.iii. It follows that $\sigma_2 = -F + G + (\ell + 1)e$, $p_1 = (\ell + 1)F + 2e$; then $u = \ell + 1 - 2 \geq 2$ and $\alpha > 0$.

- Case $\sigma_2 - p_3$: In this case $u \geq 0$ and, according to Lemma 4.3.iii, we have $b \in [-1, 0]$ that implies $h \geq 0$. Thus we always have $\alpha \geq 0$, except when $u = h = 0$; it is easy to see that $u = 0$ implies $\sigma_2 = n_1 + n_2$. If $b = 0$, then $a + b = 0$ implies $p_3 = n_1 + n_2 \in \Gamma''$.

- Case $\sigma_3 - p_1 = aF + bG + ue = \alpha e$, $\sigma_3 = a'F + b'G + \nu'e$, $p_1 = a''F + \delta e$:

- if $h \geq \ell - 2 \implies \alpha > 0$, since $u \in [3 - \ell, \ell + 1]$.

- if $h = \ell - 3 = -u$, we have $\alpha = 0$ and $\nu' - \delta = 3 - \ell$, i.e. $\delta = \ell + \nu' - 3$, that implies $\nu' = 3$ and $\delta = \ell$, because $\nu' \leq 3$ and $\ell \geq \delta$; thus $p_1 = \ell n_1 \in \Gamma''$.

- if $h < \ell - 3$ and ℓ is odd, the possible cases are (1), (4) of Lemma 4.3.iii:

In case (1) of Lemma 4.3.iii, since $b \geq 1$, we get $b = 1$, thus $h = -b = \nu$, $a + b = \nu(1 + \ell)$:

- $h = -1$, $b = 1, a + b = -1 - \ell$; since $a' + b' \geq 0$, we deduce $a'' = \ell + 1$, hence $p_1 = t_1 = (\ell + 1)F + 2e$, $\sigma_3 = p'_3 + t_2$ has $\nu' = \ell + 1$, so $u = \ell - 1 > 2$ and then $\alpha = h + \ell - 1 > 0$.

In case (4) of Lemma 4.3.iii:

- $h = \ell - 4$ then $\alpha > 0$ except $u \in [4 - \ell, 3 - \ell]$.

If $u = 4 - \ell$, then $\alpha = 0$ and $\delta = \ell + \nu' - 4$, thus $\nu' \in [3, 4]$ since $\delta \leq \ell$. Hence

$\nu' = 4 \implies \delta = \ell \implies p_1 = \ell n_1 \in \Gamma''$.

$\nu' = 3 \implies \delta = \ell - 1 \implies p_1 = (\ell - 1)n_1 \in \Gamma''$, because if $p_1 = t_2$ then $a + b \in [1, 2\ell - 1]$ that is a contradiction by (4).

If $u = 3 - \ell$, then $\delta = \ell + \nu' - 3$ implies $\nu' = 3$ as above, then the possible σ_3 are $p_3 + n_1$, $p_3 + n_2$ and so $b = b' \leq \ell + 2$, that is incompatible with the value $b = \ell + 3$ in (4).

- $h < \ell - 3$ and ℓ is even, then $h = 0$, $b = -\nu$, $a + b = \nu(1 + \ell)$. Since in $\sigma_3 - p_1$ we have $a + b \geq -\ell - 1$ then $\nu \geq -1$. Now $b \geq 1$, implies $\nu = -1, b = 1$, and $a + b = -\ell - 1$.

Therefore we can proceed as in the case $h = -1, b = 1$ treated above when ℓ is odd.

- Case $\sigma_3 - p_3$: one always has $\alpha > 0$, except if $b = 1 = u, a + b = -\ell - 1$ and ℓ is odd; in this case $\sigma_3 = a'F + b'G + \nu'e$, with $a' + b' \geq 0$, hence $a + b = -\ell - 1$ implies that $p_3 = a''F + b''G + 2e$, with $a'' + b'' = \ell + 1$ and $a' + b' = 0$. Then $\sigma_3 = p'_3 + t_2 = a'F + b'G + (\ell + 1)e$ and $u = \ell - 1 > 2$, that is a contradiction because $u = 1$. This proves iii. \square

Proposition 4.5. *The Apéry set of S is*

$$\{0\} \cup \Gamma' \quad \text{with } \Gamma' = \{kn_1 \mid k \in [1, \ell]\} \cup \{n_2\} \cup \{s_{p,q}\} \cup \{r_{p,q}\} \cup \{t_1, t_2\}$$

$$Ap_2 = \{2n_1, n_1 + n_2, 2n_2\} \quad Ap_k = \{kn_1\}, \quad \text{for } 3 \leq k \leq \ell.$$

Further $\ell n_1 - e$ is the Frobenius number of S and, with the notation of Theorem 2.2,

$$A = \{0, n_2, s_{0,\ell+1}\} \cup \{kn_1 \mid k \in [1, \ell]\}, \quad B = \{s_{p,q} \mid (p, q) \neq (0, \ell + 1)\} \cup \{r_{p,q}\} \cup \{t_1, t_2\}.$$

Proof. We have that $|\{0\} \cup \Gamma'| = e$, as follows by Construction 2.6; further these elements are all distinct *mode* by the previous lemma. Now we want to prove that for $s, s_1, \dots, s_r \in \Gamma'$, the equality $s = s_1 + \dots + s_r + \beta e$, $r \geq 2$, with $\beta \geq 0$, is impossible or implies $\beta = 0$ and $s \in \Gamma'' = \{n_1 + n_2, 2n_2\} \cup \{kn_1 \mid k \geq 2\}$.

By the first assertion, there exists $s' \in \Gamma'$ such that $s_1 + s_2 = s' + \beta'e$, where $\beta' \geq 0$ by

Lemma 4.4. Therefore if $r \geq 3$, we have $s = s' + s_3 + \dots + s_r + (\beta + \beta')e$; by iterating and by Lemma 4.4.iii, we deduce that the unique possible case is $s \in \Gamma''$ with $\beta = 0$. Since the minimal generators of S are in Γ' , this proves that for every element $s \in \Gamma'$ we have $s - e \notin S$ and, by Lemma 4.4.ii this means that $Ap = 0 \cup \Gamma'$; moreover the elements in Γ' of order greater than 1 are in Γ'' .

Now we show that $Ap_2 = \{2n_1, n_1 + n_2, 2n_2\}$. In fact, recalling that, by Lemma 4.2.ii, $n_1 < n_2$ are the smallest elements in Γ' , it follows that $\text{ord}(2n_1) = \text{ord}(n_1 + n_2) = 2$. On the other hand $\text{ord}(2n_2) = 2$, because in the proof of Lemma 4.4.iii we proved that $\sigma_i = 2n_2 \iff \sigma_i = n_2 + n_2$.

Moreover $\text{ord}(kn_1) = k$ because n_1 is the smallest element of S and then $Ap_k = \{kn_1\}$.

Finally, it follows from Lemma 4.2.ii that $(\ell n_1 - e)$ is the Frobenius number of S and, by construction and by Lemma 4.2.i, we have:

$$A = \{0, n_2, s_{0,\ell+1}\} \cup \{kn_1 \mid k \in [1, \ell]\}, \quad B = \{s_{p,q} \mid q \neq \ell + 1\} \cup \{r_{p,q}\} \cup \{t_1, t_2\}. \quad \square$$

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